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Remarks on the rotation of a magnetized sphere with application to solar rotation.

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With 2 figures in the text.

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§ 1. Elementary considerations.

Suppose that a uniformly magnetized sphere (radius $= R_0$) rotates with constant angular velocity ω_0 around the axis of magnetization. Suppose further that the sphere is electrically conducting and that, with respect to a rotating coordinate system in which the sphere rests, its electrostatic potential is zero.

Transforming to a coordinate system which does not take part in the rotation we find that — seen from the fixed system — the potential of the sphere differs from zero except at the poles. The potential at the latitude φ amounts to

$$E_\varphi = \frac{1}{c}\int (\vec{H} \times \vec{v})\, \vec{ds} \tag{1.1}$$

where $ds = R_0\, d\varphi$ is the line element, $v = \omega_0 R_0 \cos\varphi$ the velocity and H the magnetic field. If a is the dipole moment of the sphere the vertical component of H amounts to

$$H_v = \frac{2\, a \sin\varphi}{R_0^3}\,.$$

and we obtain

$$E_\varphi = \frac{a\,\omega_0}{c\,R_0} \cos^2\varphi\,. \tag{1.2}$$

If no electric charge is present in the space around the sphere, the electrostatic field is zero, seen from the moving system. Consequently a charged particle (electron or ion) can remain at rest in relation to the moving system (if centrifugal force is neglected). If the particle has a small velocity it will move in a circle which is subjected to drifts, mainly because of the inhomogeneity of the magnetic field, according to elementary laws.[1] If the velocity vector is situated in the equatorial plane of the rotating sphere, the particle moves in a circle, which is subjected to a »magnetic drift». This is very small if the velocity of the particle is small (see the cited paper).

In the fixed system the electrostatic field differs from zero (except on the axis of rotation) because of the potential of the sphere. (The equipotential surfaces are obtained through rotating the magnetic lines of force around the axis of the sphere). Consequently, a charged particle cannot rest in relation to the fixed system. If initially at rest it is accelerated by the electric field. The combination of the magnetic and the electrostatic fields tends to make it move with the velocity $R\,\omega_0$, i. e. to take part in the rotation of the sphere. This is easily seen from the conditions in the rotating system, but of course it can also be obtained through calculating the motion of a particle in the electric and magnetic fields in the fixed system.

The consequence of the above is that if no considerable space charge is present *a charged particle in the space around the magnetized sphere must take part in its rotation.*

These elementary considerations can be applied to the rotation of the sun (and the earth). Suppose that the sun's general magnetic field is a dipole field, its axis coinciding with the axis of rotation, and that the sun is electrically neutral with respect to a system rotating with the sun (taking no account of the non-uniform rotation). Suppose further that in the space around the sun, the number of charged particles is so small that they do not cause appreciable electric fields. Under these conditions *all the charged particles must take part in the solar rotation.* (We may regard them as rigidly connected with the sun by means of the magnetic »lines of force».) This does not only apply to the space close to the sun (for example the solar corona) but also — more or less — to the whole planetary system.

The outer limit for the validity depends upon how far the assumption is valid that no disturbing electric fields are pre-

[1] H. ALFVÉN: On the motion of a charged particle, Ark. f. Mat., Astr. och Fysik, Bd 27 A, N:o 22, 1940.

sent. Another outer limit is set by the fact that the centrifugal force affects the motion. An electron or ion (charge $= e$, mass $= m$) moving in the equatorial plane at the distance R from the sun is subject to a centrifugal force

$$f_c = m R \omega_0^2$$

if it takes part in the solar rotation. This force must be much smaller than the force

$$e F = \frac{e}{c} \frac{a \omega_0}{R^2}$$

due to the sun's electric polarisation in order not to disturb the motion of the charged particle.

The condition for this is

$$R \ll \left(\frac{e}{c} \frac{a}{m \omega_0}\right)^{1/3} \tag{1.3}$$

Putting $a = 4.2 \cdot 10^{33}$ gauss cm^3, $\omega_0 = \dfrac{2\pi}{25 \cdot 0.864 \cdot 10^5} = 2.9 \cdot 10^{-6}$ sec^{-1} (sun's dipole moment and angular velocity) we obtain for a proton ($e = 4.8 \cdot 10^{-10}$, $m = 1.66 \cdot 10^{-24}$ g)

$$R \ll 2.4 \cdot 10^{14} \text{ cm}$$

and for an electron

$$R \ll 2.9 \cdot 10^{15} \text{ cm}$$

(Compare: orbital radius of earth $= 1.5 \cdot 10^{13}$ cm).

§ 2. Electrically conducting matter in the magnetic field.

Suppose that in the magnetic field from the rotating magnetized sphere (S) there is situated a piece of electrically conducting matter BB', which does not take part in the rotation. (Example: a cloud of ionized gas in the sun's magnetic field). Suppose further that in the space around the sphere and the conductor BB' the density of matter is very small and that there are plenty of charged particles (electrons and ions) at the surfaces of the sphere as well as of the conductor BB'. If such particles move under the influence of the magnetic field they spiral around the magnetic lines of force. We assume that the radius of curvature of the spiral is very small compared with the distance to the centre of the sphere. (This

condition is very well satisfied in the sun's magnetic field, at least for thermal energies.)

Under these assumptions an electric field in the direction of the magnetic field will in general produce an electric current in the direction of the field, whereas an electric field perpendicular to the magnetic field gives no current in the field direction. In other words, the electrical conductivity is much larger parallel to the magnetic field than perpendicular to it.[1] (We can regard the »magnetic lines of force» as electrically conducting wires, insulated one from the other.)

Seen from a fixed coordinate system the conductivity of the stationary conductor BB' will tend to equalize the electric potentials at B and at B'. On the other hand, the rotation of the magnetized sphere produces a potential difference $E_{AA'}$, between the points A and A' where the magnetic lines of force through B and B' intersect the sphere. According to (1.2) we have

$$E_{AA'} = \frac{a\,\omega_0}{c\,R_0}(\cos^2\varphi_A - \cos^2\varphi_{A'}) \tag{2.1}$$

As charged particles can move along the magnetic line of force between A and B and also between A' and B', the e. m. f. given by (2.1) will produce a current in the circuit $A\,B\,B'\,A'\,A$.

The same result is of course obtained if the problem is treated in a coordinate system rotating with the magnetized sphere. In this case no e. m. f. is produced between A and A', but the conductor BB' is polarized, due to its motion (in relation to this system). If the conductor is situated in the equatorial plane the difference in potential between B and B' is

$$E_{BB'} = \frac{1}{c}\int\limits_{B}^{B'} (\vec{H} \times \vec{v})\,d\vec{s} = \frac{1}{c}\int\limits_{B}^{B'} \frac{a}{R^3}\cdot \omega_0\,R\,d\,R =$$

$$= \frac{a\,\omega_0}{c}\left(\frac{1}{R_B} - \frac{1}{R_{B'}}\right) \tag{2.2}$$

which is the same as (2.1) because the equation of the magnetic lines of force from a dipole gives

[1] Problems of this kind have been treated in a series of papers: H. ALFVÉN; loc. cit.; A theory of magnetic storms, Kungl. Vet.-Ak:s Handl. III, Bd 18, N:o 3, 1939; Ib. Bd 18, N:o 9, 1940; Solar prominences, Ark. f Mat., Astr. o. Fysik, Bd 27 A, N:o 20, 1940; On the solar corona, Ark. f. Mat., Astr. o. Fysik, Bd 27 A, N:o 25, 1941.

$$\frac{\cos^2 \varphi}{R_0} = \frac{1}{R} \tag{2.3}$$

(The analysis has been carried out in detail because it is very easy to make slips in a discussion of this kind.)

The electromotive force (2.1) or (2.2) will produce a current in the circuit $ABB'A'A$. The effect of the magnetic field upon this current is to speed up the conductor BB' and slow down AA'. Suppose that the current amounts to I e. m. u. The force accelerating BB' is

$$f_1 = I \int_{B}^{B'} H\,dr = Ia \left(\frac{1}{R_B^2} - \frac{1}{R_{B'}^2} \right) \tag{2.4}$$

At the same time the force

$$f_2 = I \int_{\varphi_{A'}}^{\varphi_A} H_r R_0 \, d\varphi = \frac{2Ia}{R_0^2} \int_{A}^{A'} \sin \varphi \, d\varphi =$$

$$= \frac{2Ia}{R_0^2} (\cos \varphi_A - \cos \varphi_{A'}) \tag{2.5}$$

acts upon AA'. It is easily seen that the moments about the axis of rotation of f_1 and of f_2 are equal.

§ 3. Application to the solar rotation.

The mechanism above discussed may be of some importance in connection with the rotation of the sun, if the current I is

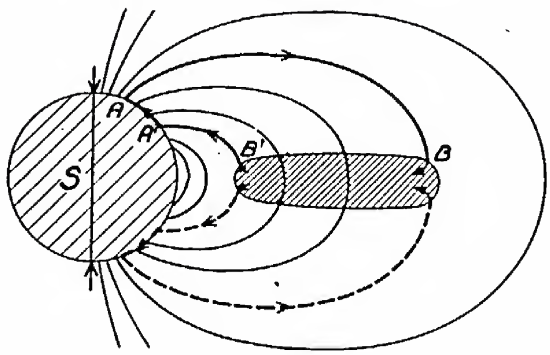

Fig. 1.

large. Therefore it is interesting to try to estimate the upper limit to I. Introducing into (3) the values $a = 4.2 \cdot 10^{33}$ gauss cm^3, $\omega_0 = 2.6 \cdot 10^{-6}$ sec^{-1}, $R_0 = 7 \cdot 10^{10}$ cm (see § 1) the factor before the parentheses becomes equal to $5.8 \cdot 10^6$ e. s. u. or $1.7 \cdot 10^9$ volts. Consequently an e. m. f. of this order of magnitude will cause something like a gaseous discharge between A and B and between A' and B'. If the conductor BB' is an ion cloud of considerable density, there are enough ions and electrons at BB' as well as at AA' (at the surface of sun) so that probably the discharge current is not limited because of a lack of charged particles. As particles of both signs are present, space charge limitation is probably of little importance. However a limit to the current is reached when the magnetic field produced by the current itself considerably disturbs the magnetic field which produces the e. m. f. This limit is set by

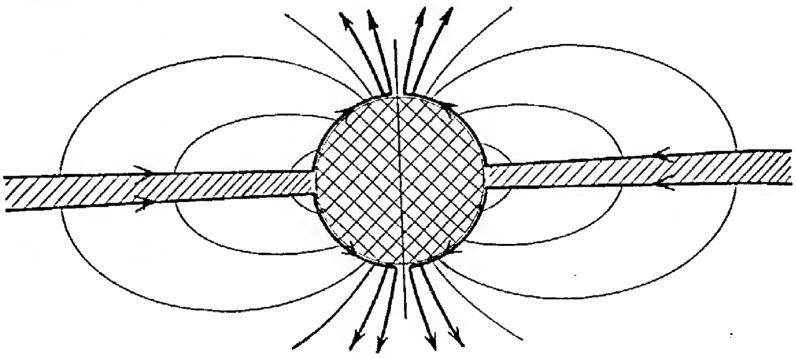

Fig. 2.

$$H \sim \frac{I_{\max}}{R_0} \sim \frac{a}{R_0^3} \tag{3.1}$$

which gives

$$I_{\max} \sim \frac{a}{R_0^2} \approx 10^{12} \cdot \text{e. m. u.} = 10^{13} \text{ amp.}$$

As it is difficult to imagine any other process limiting the current at a lower value it is reasonable to suppose that it reaches this order of magnitude. Then the current density is of the order of $I_{\max} R_0^{-2} = 10^{-9}$ amp cm^{-2}, which can be transported by electrons having a space density of $\frac{I_{\max}}{R_0^2 ec}$ which is less than one electron per cm^3. Our discussion is valid if the density of the ion cloud BB' is much greater than this.

In order to estimate the mechanical forces produced by such a current, let us consider the simple case when the current enters the sphere at the equator, flows northwards and southwards along the meridians and leaves it at the two poles (Fig. 2). Such a current system is produced if the ion cloud has the shape of a disc in the equatorial plane and reaches from the sphere to infinity. Further, the voltage drop (due to finite conductivity) must satisfy certain conditions in order to nullify potential differences, and consequently currents, along the magnetic lines of force — except at the poles.

If the total current amounts to $2\,I$, a parallel circle at the latitude φ is crossed by the current I. The mechanical force acting upon the line element $R_0\,d\,\varphi$ amounts to

$$f = \frac{2\,I\,a}{R_0^2}\sin\varphi\,d\,\varphi \tag{3.2}$$

according to (2.5). Let us assume that this force retards the rotation of a layer of the sun reckoned from the surface down to the depth D below the surface. The mass of a ring with cross-section $D \cdot R_0\,d\,\varphi$ is $2\,\pi\,\varrho\,R_0^2\,D\,\cos\varphi\,d\,\varphi$ (where ϱ is the density). If its velocity is changed at the rate $R_0\cos\varphi\,\frac{d\,\omega}{d\,t}$ we have

$$2\,\pi\,\varrho\,R_0^2\,D\cos\varphi\,d\,\varphi \cdot R_0\cos\varphi\,\frac{d\,\omega}{d\,t} = f = \frac{2\,I\,a}{R_0^2}\sin\varphi\,d\,\varphi$$

or

$$\frac{d\,\omega}{d\,t} = \frac{I}{\pi\,\varrho}\,\frac{a}{R_0^5\,D}\,\frac{\sin\varphi}{\cos^2\varphi}. \tag{3.3}$$

Putting $\varrho = 1$, $D = 0.1\,R_0$ and $I = 10^{12}$ e. m. u. we obtain

$$\frac{d\,\omega}{d\,t} = 1.1 \cdot 10^{-19}\,\frac{\sin\varphi}{\cos^2\varphi}. \tag{3.4}$$

At $\varphi = 30°$, ω is changed by 10 % of its present value ($\omega_0 = 2.9 \cdot 10^{-6}$) after

$$t = 4 \cdot 10^{12}\ \text{sec} \approx 10^5\ \text{years}.$$

As this length of time is very small in comparison to the age of the sun, we conclude that *the mechanical forces from our currents are strong enough to affect the solar speed of rotation very much.* In other words, *an ion cloud in the neighbourhood*

of the sun is able to retard solar rotation through the electromagnetic effect considered.

Of course the mass of the cloud must be so large that it can take up the angular momentum. From (2.4) it is easily seen that in the case which we have discussed, half of the angular momentum must be taken up by that part of the cloud which is situated between R_0 and $2R_0$, i. e. below one solar radius from the surface. However, as initially the cloud must be at rest and consequently of extra-solar origin, it seems very unlikely that it could have such a density distribution that so much angular momentum could be taken up by a part so close to the sun's surface. It is more probable that most of the momentum is taken up by parts more distant from the sun. If so, currents must flow along magnetic lines of force from the equatorial plane to the sphere, so that $I(\lambda)$ increases with the latitude.

Equation (3.4) shows that even if $I(\lambda) =$ const., the rate of retardation increases with the latitude.[1] The increase is still larger if I increases with the latitude, as it is likely to do. Consequently we must expect this process to slow down the solar rotation the more, the higher is the latitude. This is very satisfactory because *the sun does rotate slower at high latitudes than at low latitudes.* Thus *the non-uniform rotation of the sun may be explained as a result of an electromagnetic retardation.* At the present time the process may go on if there is a continuous inflow of ionized gas from interstellar space into the neighbourhood of the sun. However, it is more probable that such a phenomenon occurred a long time ago (maybe in connection with the genesis of the planetary system), and that the viscosity of the sun has not yet been able to equalize the rotation.

Jupiter and Saturn also exhibit non-uniform rotations. As it is likely that they possess magnetic fields — as the earth and the sun do — these phenomena may also be due to electromagnetic retardation.

Summary.

If an electrically conducting magnetized body rotates, electrically charged particles in its environment will have a tendency to take part in the rotation.

[1] The function becomes infinite for $\varphi = \frac{\pi}{2}$ only because we have supposed that the whole current is concentrated to the axis, which of course it cannot be.

If a number of charged particles, e. g. an ion cloud, is initially at rest in the neighbourhood of the body, a system of currents is produced which accelerates the particles and retards the rotation of the body, thus equalizing their angular velocities.

It is shown that this effect may be of importance in the solar rotation. If an ion cloud invades the magnetic field of the sun, a considerable part of the rotational moment of the sun can be transferred to the cloud within a reasonable length of time. The retardation of the sun's rotation is strongest at high latitudes.

The fact that the sun does rotate more slowly at high latitudes may indicate that a process of this kind is going on or, more likely, has once occurred, perhaps in connection with the genesis of the planetary system.

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